现代控制理论 Modern Control Theory

Introduction

Essential Prerequisites

1、 Elementary Circuit Theory

2、 Elementary Control Theory

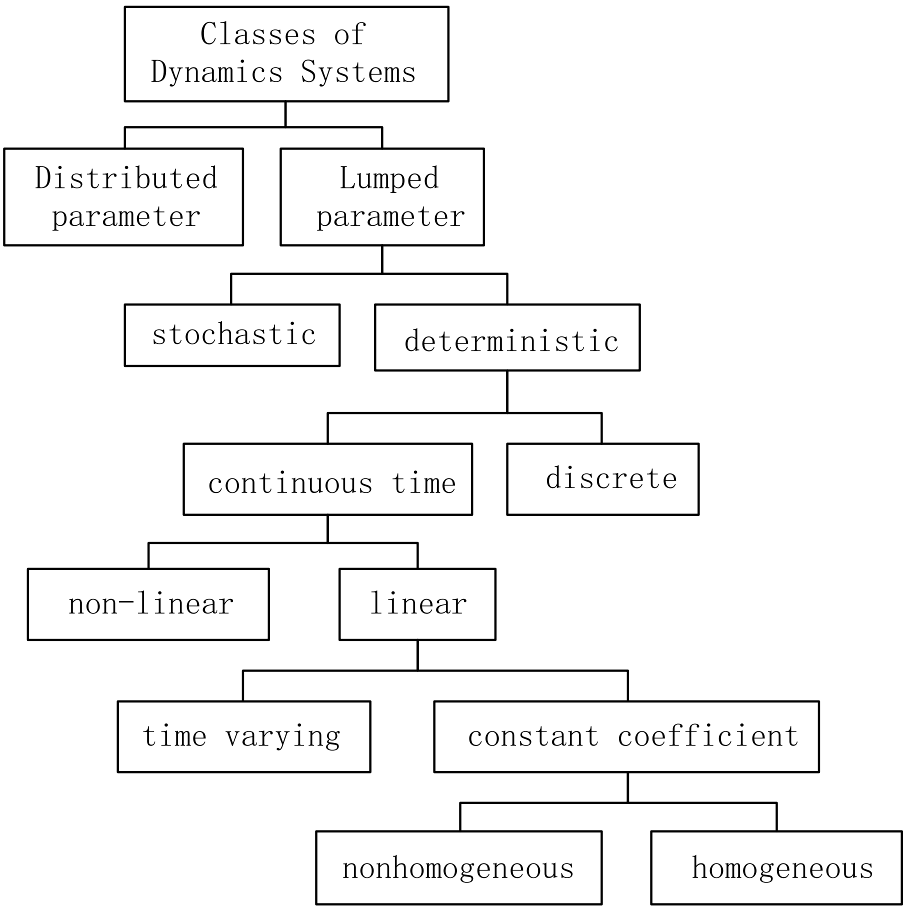
3、 Matrix and its Manipulation

Classic Control Theory: single input-single output linear constant system (SISO)
Variable: input, output, error—feedback system
Method: frequency domain (response 响应)
root locus

Modern Control Theory: multi-input—multi-output
Constant or time-varying, linear or non-linear system
Method: state-space (状态空间) ， time-domain

Dynamic—differential equation

Static—algebraic equation



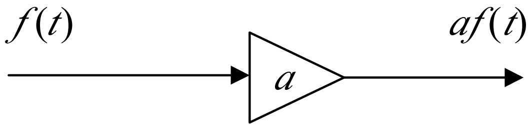
Chapter I state space method in linear system

Part 1 Continuous System

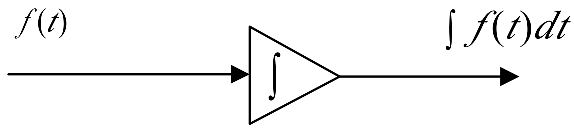
Section 1 The State Space Prescription Of Linear Time-invariant systems

1. Linear Components on Analog Simulation

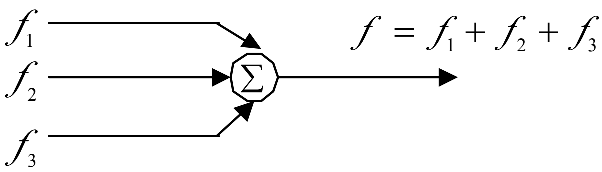
Amplifier



Integrator

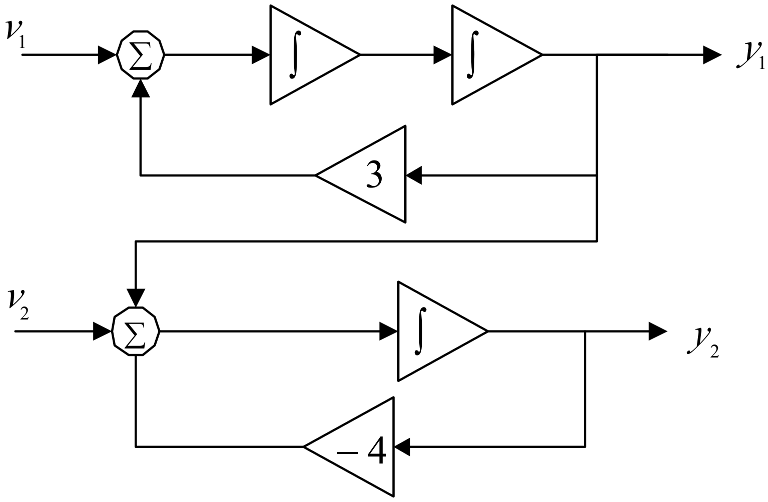


Summer



2. Formulation of State Equations

 2.1 From Simulation Diagram



state variable: output of integrator

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 3x_1 + v_1 \\ \dot{x}_3 = x_1 - 4x_3 + v_2 \\ y_1 = x_1 \\ y_2 = x_3 \end{cases}$$

即 $$\begin{cases} \dot{X} = AX + BV \\ Y = CX + DV \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

2.2 From Differential Equation

$$\ddot{y} + 7\dot{y} + 6y = 4v$$

define $\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases}$

$$\begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = -7\dot{y} - 6y + 4v = -6x_1 - 7x_2 + 4v \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] v$$

$$\ddot{y} + 7\dot{y} + 6y = 4v + 8\dot{v}$$

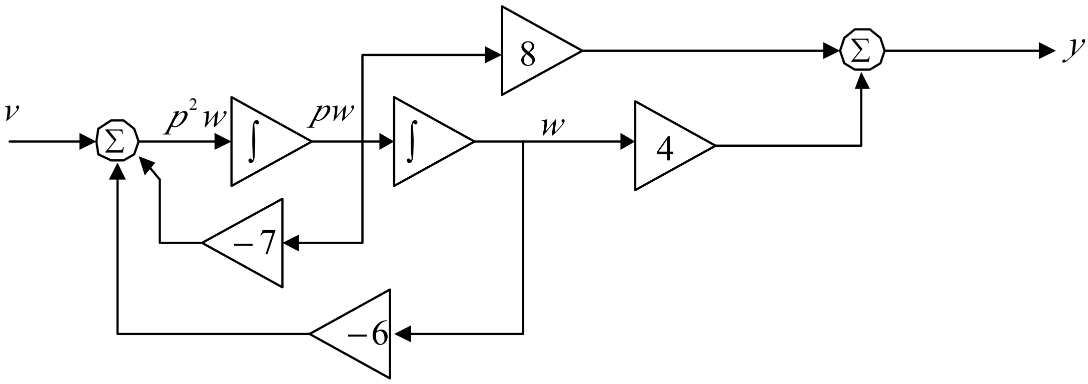
method 1. use operator $p = \frac{d}{dt}$

$$p^2 y + 7py + 6y = 4v + 8pv$$

$(p^2 + 7p + 6)y = (8p + 4)v$ here $w(t)$ is auxiliary variable

$$\frac{y(t)}{v(t)} = \frac{8p+4}{p^2 + 7p + 6} \frac{w(t)}{w(t)}$$

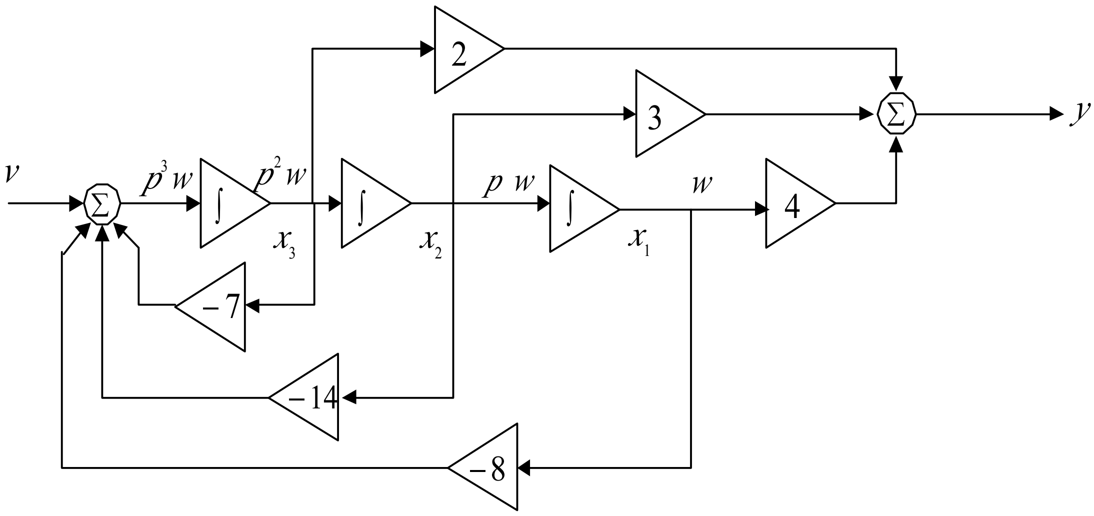
use simulation diagram as a tool



$$\dddot{y} + 7\ddot{y} + 14\dot{y} + 8y = 2\ddot{v} + 3\dot{v} + 4v$$

$$(p^3 + 7p^2 + 14p + 8)y = (2p^2 + 3p + 4)v$$

$$\frac{y(t)}{v(t)} = \frac{2p^2 + 3p + 4}{p^3 + 7p^2 + 14p + 8}\frac{w(t)}{w(t)}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v$$

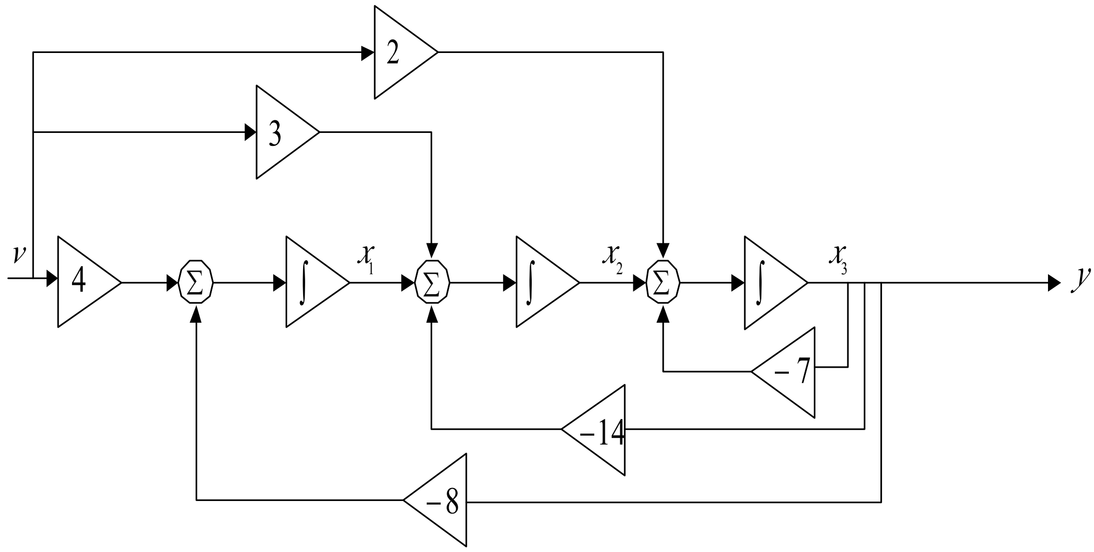
$$y = \begin{bmatrix} 4 & 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]v$$

method 2. ($\frac{1}{p}$ nested form)

$$p^3 y + 7p^2 y + 14py + 8y = 2p^2 v + 3pv + 4v$$

$$p^3 y = p^2(-7y + 2v) + p(-14y + 3v) + (-8y + 4v)$$

$$y = \frac{1}{p}\{-7y + 2v + \frac{1}{p}[-14y + 3v + \frac{1}{p}(-8y + 4v)]\}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -8 \\ 1 & 0 & -14 \\ 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} v$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]v$$

method 3. diagonalized, decouple

$$\begin{cases} \dot{X} = AX + BV \\ Y = CX + DV \end{cases}$$

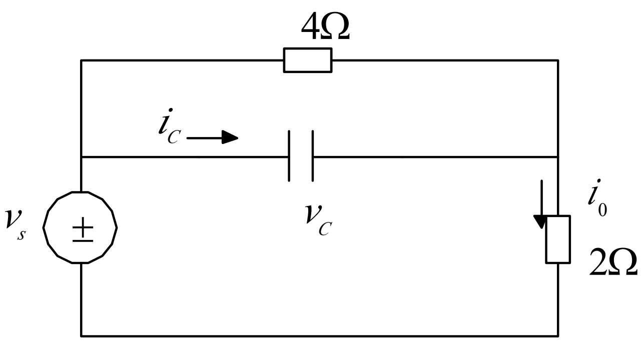
set an auxiliary variable q

$$x = Mq \qquad q = M^{-1}x \quad (M \neq 0)$$

$$\frac{d}{dt}Mq = M\dot{q} = AMq + BV$$

$$\dot{q} = M^{-1}AMq + M^{-1}BV \qquad M^{-1}AM \quad \text{diagonal}$$
$$y = CMq + DV$$

2.3 From Circuit Diagram



state variable
$$\begin{cases} i_C = c\dfrac{du_C}{dt} \\ u_L = L\dfrac{di_L}{dt} \end{cases}$$

$$X = v_C$$

input v_S

output i_0

let c=5

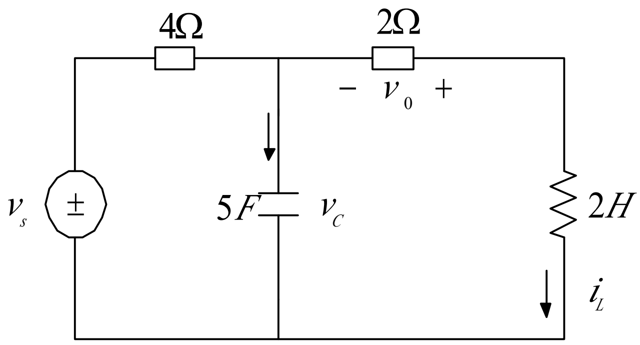
$$i_C = c\dot{v}_C = 5\dot{v}_C$$

$$\dot{v}_C = \frac{1}{5}i_C$$

$$i_C + \frac{1}{4}v_C = i_0 = \frac{v_S - v_C}{2}$$

$$i_C = \frac{1}{2}v_S - \frac{3}{4}v_C$$

$$\begin{cases} \dot{v}_C = -\dfrac{3}{20}v_C + \dfrac{1}{10}v_S \\ i_0 = -\dfrac{1}{2}v_C + \dfrac{1}{2}v_S \end{cases}$$



$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{20} & -\frac{1}{5} \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{20} \\ 0 \end{bmatrix} v_S$$

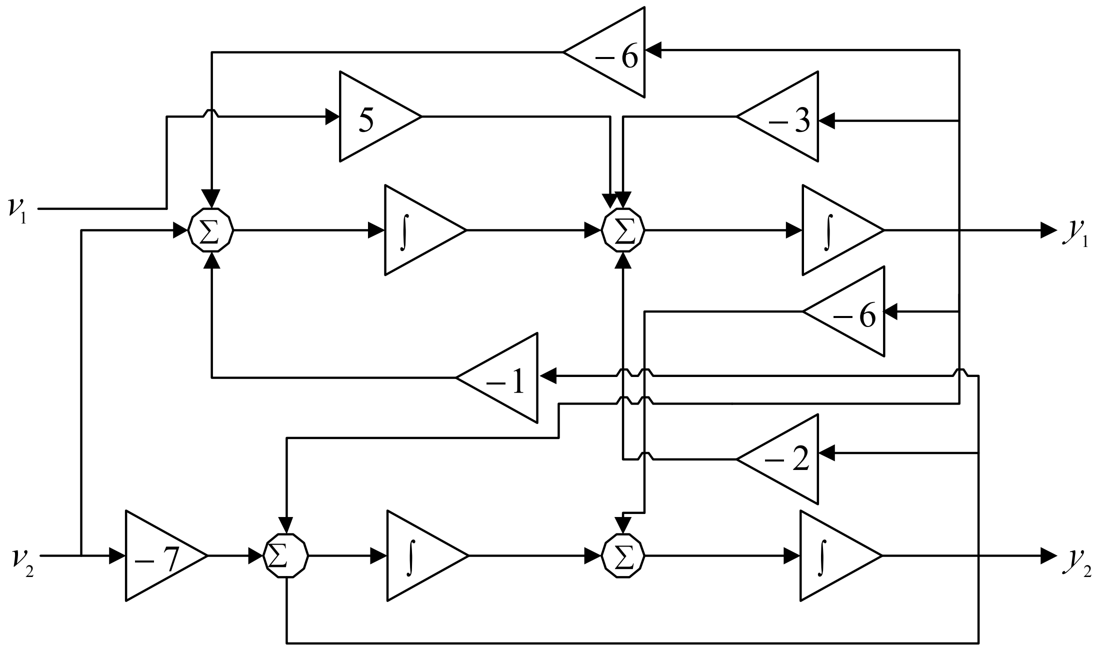
$$y = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + [0] v_S$$

2.4 Simultaneous Differential Equation

$$\begin{cases} \ddot{y}_1 + 3\dot{y}_1 + 6y_1 + 2\dot{y}_2 + y_2 = 5\dot{v}_1 + v_2 \\ \ddot{y}_2 + 6\dot{y}_1 - y_1 - y_2 = 7v_2 \end{cases}$$

nested form $\begin{cases} p^2 y_1 = -3py_1 - 6y_1 - 2py_2 - y_2 + 5pv_1 + v_2 \\ p^2 y_2 = -6py_1 + y_1 + y_2 + 7v_2 \end{cases}$

$$\begin{cases} y_1 = \frac{1}{p}[-3y_1 - 2y_2 + 5v_1 + \frac{1}{p}(-6y_1 - y_2 + v_2)] \\ y_2 = \frac{1}{p}[-6y_1 + \frac{1}{p}(y_1 + y_2 + 7v_2)] \end{cases}$$



2.5 From Transfer Function

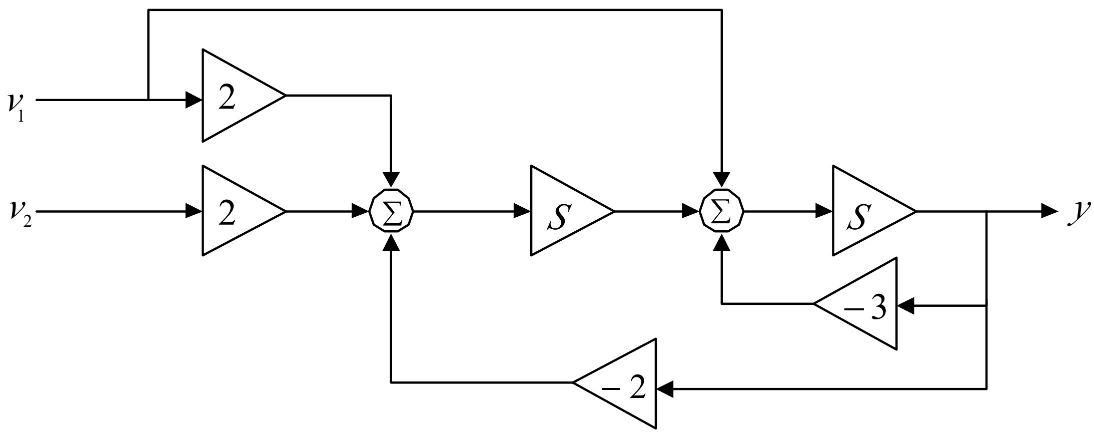
1 output y

2 input v_1, v_2

$$y=\begin{bmatrix}\frac{1}{s} & \frac{2}{(s+1)(s+2)}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\frac{v_1}{s+1}+\frac{2v_2}{(s+1)(s+2)}$$

$$(s^2+3s+2)y=(s+2)v_1+2v_2$$

$$y=\frac{1}{s}\{-3y+v_1+\frac{1}{s}[-2y+2v_1+2v_2]\}$$



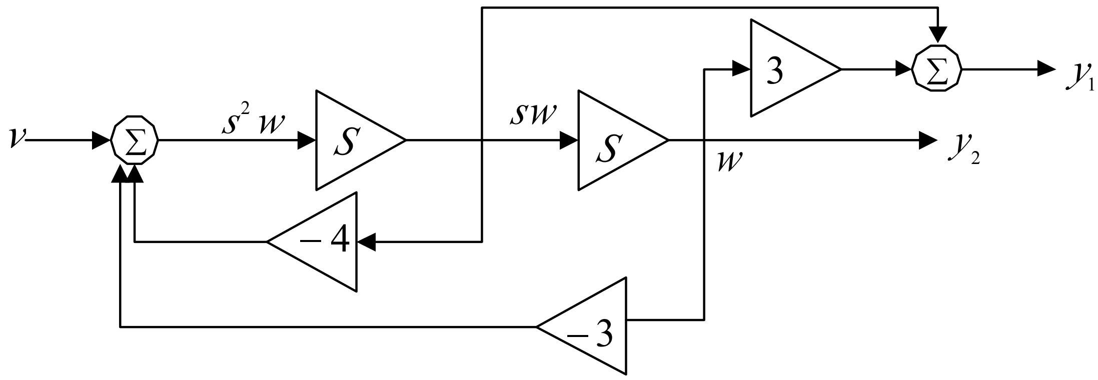
2 output y_1, y_2

1 input v

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{(s+1)(s+3)} \end{bmatrix} v$$

$$\begin{cases} y_1 = \dfrac{1}{s+1} v = (s+3)\,w \\ y_2 = \dfrac{1}{(s+1)(s+3)} v = w \end{cases}$$ auxiliary variable w

$$(s+1)(s+3)\,w = v = (s^2 + 4s + 3)\,w$$



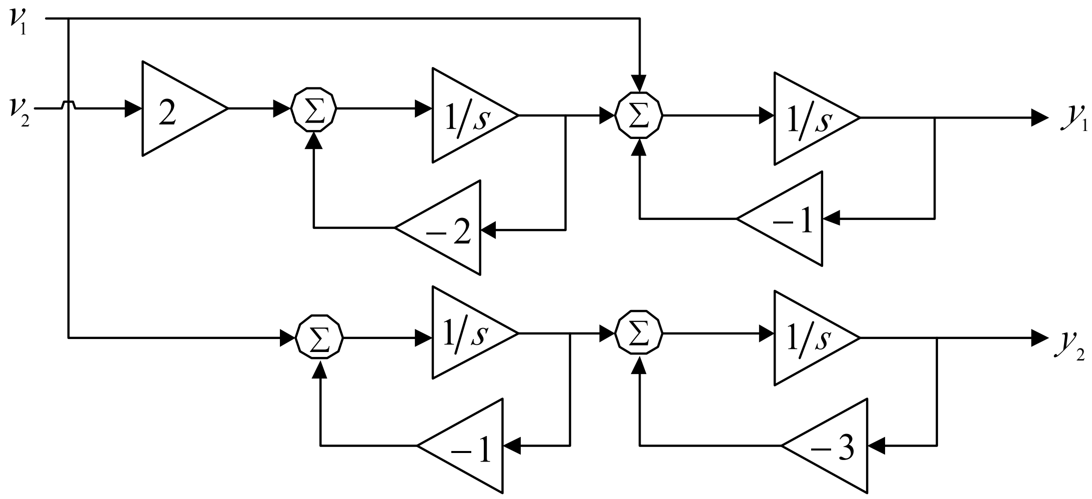
2 input and 2 output

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+3)} & \dfrac{1}{s+3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

solution 1: treat as 2 problems

$$y_1 = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} \dfrac{1}{(s+1)(s+3)} & \dfrac{1}{s+3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



solution 2: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{(s+1)(s+3)} \end{bmatrix} v_1 + \begin{bmatrix} \dfrac{2}{(s+1)(s+2)} \\ \dfrac{1}{s+3} \end{bmatrix} v_2$

section 2 The Solution of State Equation Of Linear Time-invariant System

Given $\begin{cases} \dot{X} = AX + BV \\ Y = CX + DV \end{cases}$ A、B、C、D are time invariant or constant coefficient

Initial condition $X(0)$ $V(t)$ for $t \geq 0$

Find $y(t)$ for $t \geq 0$

Solution 1. Frequency Domain Method

2. Time Domain Method

1. Frequency Domain Method by Laplace Transformation

$$SX - X(0) = AX + BV$$
$$(SI - A)X = BV + X(0)$$
$$X = [SI - A]^{-1} BV + [SI - A]^{-1} X(0)$$
$$Y = C[SI - A]^{-1} BV + C[SI - A]^{-1} X(0) + DV$$
$$= C[SI - A]^{-1} X(0) + \{C[SI - A]^{-1} B + D\} V$$

2. Time Domain Method by the infinite series

Consider unforced system $\begin{matrix} \dot{X} = AX \\ Y = CX \end{matrix}$ $X(0)$

Consider the special case (scalar state) $\dot{x} = ax$, $x(0)$

And the solution is well-known

$$x(t) = e^{at}k$$
$$x(0) = e^{0}k$$
$$x(t) = x(0)e^{at}$$

In general vector case $\dot{X} = AX$, $X(0)$

The solution is $X(t) = e^{At} \cdot X(0)$

How to find Matrix exponential e^{At} here A $n \times n$ Matrix

define
$$\begin{cases} e^{At} \overset{\Delta}{=} 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots \\ e^{At} \overset{\Delta}{=} I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \end{cases}$$

$$\begin{cases} e^{0t} \overset{\Delta}{=} e^{a\cdot 0} = 1 \\ e^{[0]t} \overset{\Delta}{=} e^{A\cdot 0} = I_n \end{cases}$$

$$\begin{cases} e^{at} \cdot e^{bt} = e^{(a+b)t} \\ e^{At} \cdot e^{Bt} = e^{(A+B)t} \end{cases} \quad \text{iff. } AB = BA$$

$$\begin{cases} [e^{at}]^{-1} = e^{-at} \\ [e^{At}]^{-1} = e^{-At} \end{cases}$$

proof $e^{At} \cdot e^{-At} = I$

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At} \cdot A$$

proof $\frac{d}{dt}e^{At} = \frac{d}{dt}[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$

$$= [0 + a + \frac{1}{1!}A^2 t + \frac{1}{2!}A^3 t^2 + \cdots]$$

$$= A[I + At + \frac{1}{2!}A^2 t^2 + \cdots]$$

$$= Ae^{At}$$

$$\dot{X}(t) = \frac{d}{dt}X = \frac{d}{dt}(e^{At} \cdot K) = A \cdot e^{At} \cdot K = AX \quad (e^{At} \text{ is matrix})$$

example 1. A is in diagonal form

Given $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ find e^{At}

Solution $A^2 = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$ $A^3 = \begin{bmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{bmatrix}$ $\cdots$

$$A^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2!}\lambda_1^2 t^2 & 0 \\ 0 & \frac{1}{2!}\lambda_2^2 t^2 \end{bmatrix} + \cdots + \begin{bmatrix} \frac{1}{n!}\lambda_1^n t^n & 0 \\ 0 & \frac{1}{n!}\lambda_2^n t^n \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2!}\lambda_1^2 t^2 + \cdots + \frac{1}{n!}\lambda_1^n t^n & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2!}\lambda_2^2 t^2 + \cdots + \frac{1}{n!}\lambda_2^n t^n \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

example 2 Jordan Form

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{find } e^{At}$$

decouple the matrix

$$A=\begin{bmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_2 & 1\\ 0 & 0 & \lambda_3\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}+\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$A=\Lambda+P$$

$$e^{(\Lambda+P)t}=e^{\Lambda t}\cdot e^{Pt} \qquad \because \Lambda P=P\Lambda$$

$$=\begin{bmatrix}e^{\lambda_1 t} & 0 & 0\\ 0 & e^{\lambda_2 t} & 0\\ 0 & 0 & e^{\lambda_3 t}\end{bmatrix}\cdot e^{Pt}$$

$$e^{Pt}=I+Pt+\frac{1}{2!}P^2t^2+\frac{1}{3!}P^3t^3+\cdots$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}t+\frac{1}{2!}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}t^2+\frac{1}{3!}[0]t^3+\cdots$$

$$e^{At}=e^{\lambda_1 t}\cdot\begin{bmatrix}1 & t & \frac{1}{2!}t^2\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix} \qquad e^{Pt}=\begin{bmatrix}1 & t & \frac{1}{2!}t^2\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix}$$

if $A=\begin{bmatrix}-4 & 1 & 0\\ 0 & -4 & 1\\ 0 & 0 & -4\end{bmatrix}$ then $e^{At}=e^{-4t}\begin{bmatrix}1 & t & \frac{1}{2!}t^2\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix}$

Frequency Domain

$A=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$ find e^{At}

solution $[SI-A]=\begin{bmatrix}s & -1\\ 2 & S+3\end{bmatrix}$

$$[SI-A]^{-1}=\frac{1}{S(S+3)+2}\begin{bmatrix}S+3 & 1\\ -2 & S\end{bmatrix}$$

[1,1] term $\dfrac{S+3}{S^2+3S+2} = \dfrac{2}{S+1} + \dfrac{-1}{S+2} \Rightarrow 2e^{-t} - e^{-2t}$

3. Calay-Hamilton Theorem

   Matrix $A_{n\times n}$

   Characteristic Polynomial of A

   $$|\lambda I - A| = \lambda^n + C_{n-1}\lambda^{n-1} + C_{n-2}\lambda^{n-2} + \cdots + C_1\lambda + C_0 = 0$$

   i.e. $(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)\cdots(\lambda-\lambda_n) = 0$

   $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ eigenvalues of A matrix

C-H Theorem

Every square matrix satisfies its own characteristic equation.

Example $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \lambda(\lambda+3) + 2 = (\lambda+1)(\lambda+2) = 0$$

$$A^2 + 3A + 2I = 0$$

$$\begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + 2\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

every $f(\lambda)$ can be expressed in terms of $\alpha_0 + \alpha_1\lambda$

under the conditon $\lambda^2 + 3\lambda + 2 = 0$ and if $f(\lambda) = \lambda^4 + 2\lambda^3 + 1$

$$\lambda^2 = -3\lambda - 2$$

$$\lambda^3 = -3\lambda^2 - 2\lambda = -3(-3\lambda - 2) - 2\lambda = 7\lambda + 6$$

by C-H Theorem

$$A^2 + 3A + 2I = 0$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \alpha_0(t)I + \alpha_1(t)A$$

for $\lambda = -1 \qquad e^{-t} = \alpha_0(t) - \alpha_1(t)$

for $\lambda = -2 \qquad e^{-t} = \alpha_0(t) - 2\alpha_1(t)$

solution $\begin{aligned} \alpha_1(t) &= e^{-t} - e^{-2t} \\ \alpha_0(t) &= 2e^{-t} - e^{-2t} \end{aligned}$

$$e^{At} = (2e^{-t} - e^{-2t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t})\begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{vmatrix}$$

4. Diagonalization of A through similarity transformation

Unforced system $\begin{cases} \dot{X} = AX \\ X(0) \end{cases}$

Let $X = Mq \quad M$ nonsingular

Then $M\dot{q} = AMq$

$\dot{q} = M^{-1}AMq$

case 1. A has distinct eigenvalues

There must be a non-singular M such that

$$M^{-1}AM = diag(\lambda_{1,}\lambda_{2,}\lambda_3 \cdots \lambda_n) = \begin{vmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{vmatrix} = \Lambda$$

Procedure to get M

$$M = [M_1, M_2 \cdots M_n] \qquad M^{-1}AM = \Lambda \qquad AM = M\Lambda$$

$$A[M_1, M_2 \cdots M_n] = [M_1, M_2 \cdots M_n]\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$[\lambda_i I - A]M_i = 0$$

$$i = 1,2,\cdots n$$

example $$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^3 - 2\lambda^2 - 5\lambda + 6 = (\lambda + 1)(\lambda + 2)(\lambda - 3)$$

$$\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = 3$$

$$[\lambda_1 I - A]M_1 = 0$$

$$\begin{bmatrix} \lambda_1 - 2 & 2 & -3 \\ -1 & \lambda_1 - 1 & -1 \\ -1 & -3 & \lambda_1 + 1 \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = 0$$

$$\lambda_1 = 1$$

$$M_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

similarly $$M_2 = \begin{bmatrix} 11 \\ 1 \\ -14 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix} \qquad M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Given $\dot{X} = AX \quad X(0) = \begin{bmatrix} 5 \\ 6 \\ -2 \end{bmatrix}$

Find $X(t)$ for $t \geq 0$

Solution Let $X = Mq$ $\quad \dot{q} = M^{-1}AMq$

Then $\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$

$$\dot{q}_1 = q_1 \qquad q_1(t) = e^{t} q_1(0)$$
$$\dot{q}_2 = -2q_2 \qquad q_2(t) = e^{-2t} q_2(0)$$
$$\dot{q}_3 = 3q_3 \qquad q_3(t) = e^{3t} q_3(0)$$

where $q(0) = M^{-1}X(0) = \dfrac{1}{30}\begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{95}{30} \\ \frac{8}{15} \\ \frac{23}{10} \end{bmatrix}$

deduce e^{At} given A

suppose $M^{-1}AM = \Lambda$

$$AM = M\Lambda$$
$$A = M\Lambda M^{-1}$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$= MM^{-1} + tM\Lambda M^{-1} + \frac{t^2}{2!}(M\Lambda M^{-1})(M\Lambda M^{-1}) + \cdots$$

$$= M\{I + \Lambda t + \frac{1}{2!}\Lambda^2 t^2 + \cdots\}M^{-1}$$

$$= Me^{\Lambda t}M^{-1}$$

$$e^{At} = M\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}M^{-1}$$

general solution

$$X(t) = e^{At}X(0)$$
$$X(t) = Me^{\Lambda t}M^{-1}X(0)$$

Given $\begin{cases} \dot{X} = AX \\ X(0) \end{cases}$ Assume that A has distinct eigenvalue $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$

Then there exists a non-singular $M = [M_1, M_2 \cdots M_n]$ such that

$$M^{-1}AM = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Transformation Let $A = Mq$

$$\dot{q} = M^{-1}AMq = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} q$$

$$\begin{aligned} \dot{q}_1 &= \lambda_1 q_1 & \quad q_1(t) &= e^{\lambda_1 t} q_1(0) \\ \dot{q}_2 &= \lambda_2 q_2 & \quad q_2(t) &= e^{\lambda_2 t} q_2(0) \\ &\vdots & &\vdots \\ \dot{q}_n &= \lambda_n q_n & \quad q_n(t) &= e^{\lambda_n t} q_n(0) \end{aligned}$$

$$X(t) = Mq(t) = [M_1, M_2 \cdots M_n]\begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix}$$

$$= M_1 q_1(t) + M_2 q_2(t) + \cdots M_n q_n(t)$$

$$= M_1 q_1(t) e^{\lambda_1 t} + M_2 q_2(t) e^{\lambda_2 t} + \cdots M_n q_n(t) e^{\lambda_n t}$$

$$X(0) = M_1 q(0) + M_2 q(0) + \cdots + M_n q(0)$$

where $q_1(0), q_2(0) \cdots q_n(0)$ are arbitrary constants

example : $\dot{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X$

$$[\lambda I - A] = \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$M^{-1} A M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M = [M_1, M_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Let $X = Mq$ then $X(t) = M_1 q_1(0) e^{\lambda_1 t} + M_2 q_2(0) e^{\lambda_2 t}$

$$= M_1 q_1(0) e^{-t} + M_2 q_2(0) e^{t}$$

adjust initial condition to suppress e^t mode

$$q(0) = M^{-1} X(0)$$

$$\begin{bmatrix} q_1(0) \\ q_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix}$$

$$q_2(0) = 0 \quad X_1(0) = -X_2(0) \qquad X(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} q_1(0) e^{-t}$$

Case 3

A has repeated engenvalues $|\lambda I - A| = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\cdots(\lambda - \lambda_e)^{m_e}$

suppose $(\lambda - 2)^3(\lambda + 4)^4$

$$\text{find } M^{-1}AM = \left[\begin{array}{ccc|cccc} 2 & \times & & & & & \\ & 2 & \times & & & & \\ & & 2 & & & & \\ \hline & & & -4 & \times & & \\ & & & & -4 & \times & \\ & & & & & -4 & \times \\ & & & & & & -4 \end{array}\right]$$

superdiagonal $\times$ either 1 or 0

$$\text{suppose } M^{-1}AM = \left[\begin{array}{ccc|c} 4 & 1 & & \\ & 4 & 1 & \\ & & 4 & \\ \hline & & & -5 \end{array}\right] = J$$

$$AM = MJ$$

$$A[M_1, M_2, M_3, M_4] = [M_1, M_2, M_3, M_4]J$$

$$[AM_1, AM_2, AM_3, AM_4] = [\lambda_1 M_1, M_1 + \lambda_1 M_2, M_2 + \lambda_1 M_3, \lambda_2 M_4]$$

$$\begin{cases} AM_1 = \lambda_1 M_1 \\ AM_2 = M_1 + \lambda_1 M_2 \\ AM_3 = M_2 + \lambda_1 M_3 \\ AM_4 = \lambda_2 M_4 \end{cases}$$

example Given $\dot{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} X \quad X(0) = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ find $X(t)$ for $t \geq 0$

solution $|\lambda I - A| = (\lambda - 1)^2(\lambda - 2)$

$$\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$$

$$M^{-1}AM=\begin{bmatrix}1&1&0\\0&1&0\\0&0&2\end{bmatrix} \text{ or } \begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$$

assume $A[M_1,M_2,M_3]=[M_1,M_2,M_3]\begin{bmatrix}1&1&0\\0&1&0\\0&0&2\end{bmatrix}$

$$\begin{cases}AM_1=M_1\\AM_2=M_1+M_2\\AM_3=2M_3\end{cases} \qquad \begin{cases}[I-A]M_1=0\\ [I-A]M_2=-M_1\\ [2I-A]M_3=0\end{cases}$$

$$M_1=\begin{bmatrix}0\\1\\-3\end{bmatrix} \qquad M_2=\begin{bmatrix}1\\0\\-5\end{bmatrix} \qquad M_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$M=\begin{bmatrix}0&1&0\\1&0&0\\-3&-5&1\end{bmatrix} \qquad M^{-1}=\begin{bmatrix}0&1&0\\1&0&0\\5&3&1\end{bmatrix}$$

$$M^{-1}AM=\begin{bmatrix}1&1&0\\0&1&0\\0&0&2\end{bmatrix}$$

solution 1 Let $X=Mq$ then $\dot{q}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}q$

$$q_1(t)=e^t q_1(0)+te^t q_2(0)$$
$$q_2(t)=e^t q_2(0)$$
$$q_3(t)=e^{2t} q_3(0)$$

solution 2 $\dot{X}=AX$

$$X = e^{At}X(0) = M\left[\begin{array}{cc:c} e^{t} & te^{t} & \\ & e^{t} & \\ \hdashline & & e^{2t} \end{array}\right]M^{-1}X(0)$$

Forced Systems

State Transition Matrix

Definition: linear time invariant system $\dot{X}\ AX$

The STM $\phi(t,t_0)$ is a matrix (with argument t, t_0

Such that $X(t) = \phi(t,t_0)X(t_0)$ for all t, t_0

e.x. $\phi(5,2)$ then $X(5) = \phi(5,2)X(2)$

$\phi(9,6)$ then $X(9) = \phi(9,6)X(6)$

the general solution $X(t) = e^{At}K$

at $t = t_0, X(t_0) = e^{At_0}K \quad \therefore K = e^{-At_0}X(t_0)$

$$X(t) = e^{A(t-t_0)}X(t_0)$$

$$\therefore \phi(t,t_0) = e^{A(t-t_0)}$$

$$\therefore \phi(t,0) = e^{At}$$

Solution of a forced system

Given $\begin{cases} \dot{X} = AX + BV \\ Y = CX + DV \end{cases}$ $X(0)$ i.e. $X(t_0)$ find $X(t)$ for $t \le t_0$

$$e^{-At}\dot{X} = e^{-At}AX + e^{-At}BV$$

$$e^{-At}\dot{X} - e^{-At}AX = e^{-At}BV$$

$$\frac{d}{dt}[e^{-At}X(t)] = e^{-At}BV$$

$$\int_{t_0}^{t}\frac{1}{d\tau}[e^{-A\tau}X(\tau)]d\tau = \int_{t_0}^{t}e^{-A\tau}BV(\tau)d\tau$$

$$e^{-At}X(t) - e^{-At_0}X(t_0) = \int_{t_0}^{t}e^{-A\tau}BV(\tau)d\tau$$

$$X(t) = e^{A(t-t_0)}X(t_0) + e^{At}\int_{t_0}^{t}e^{-A\tau}BV(\tau)d\tau$$

$$= e^{A(t-t_0)} X(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} BV(\tau) d\tau$$

$$y(t) = Ce^{A(t-t_0)} X(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)} BV(\tau) d\tau + DV(t)$$

Frequency Domain

Solution $Y = \{C[SI - A]^{-1} B + D\} V + C[SI - A]^{-1} X(0)$

Numerical integration

Forward Euler Method (integrate step by step)

$$\dot{x} = ax + bv$$

$$x[(k+1)T] = x(kT) + T\{ax(kT) + bv(kT)\}$$

$$= (1 + aT)x(kT) + Tbv(kT)$$

$$k = 0 \quad x(T) = (1 + aT)x(0) + Tbv(0)$$

$$k = 1 \quad x(2T) = (1 + aT)x(T) + Tbv(T)$$

Backward Euler Method

$$x[(k+1)T] = x(kT) + T\frac{dx}{dt}\Big|_{t=(k+1)T}$$

$$x[(k+1)T] = x(kT) + T\{ax[(k+1)T] + bv[(k+1)T]\}$$

$$X[(k+1)T] = [I - AT]^{-1}\{X(kT) + TBV[(k+1)T]\}$$

here $a \Rightarrow A \quad b \Rightarrow B$

Trapezoidal Method

$$X[(k+1)T] = X(kT) + T \cdot \frac{1}{2}[(\frac{dx}{dt})_{t=kT} + (\frac{dx}{dt})_{t=(k+1)T}]$$

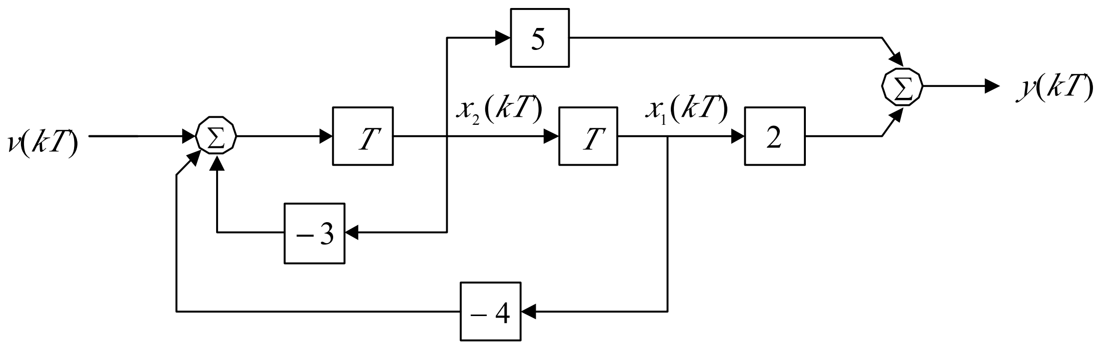
Part 2 Linear Discrete Systems

State Equation $\begin{cases} X[(k+1)T] = AX(kT) + BV(kT) \\ Y(kT) = CX(kT) + DV(kT) \end{cases}$

Formulation of State Equation

1. From Simulation Diagram
   a) amplifier
   b) summer
   c) delayer

   difference equation



$$\begin{cases} x_1[(k+1)T] = x_2(kT) \\ x_2[(k+1)T] = -4x_1(kT) - 3x_2(kT) + v(kT) \end{cases}$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

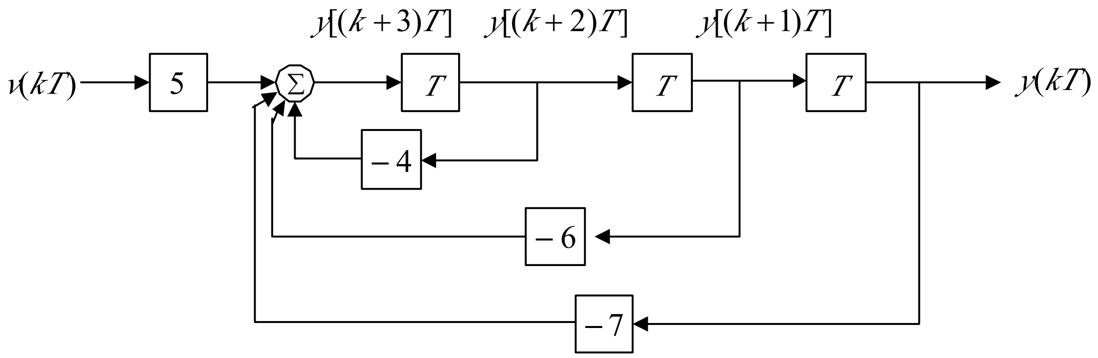
$$y(kT) = 2x_1(kT) + 5x_2(kT)$$

$$C = \begin{bmatrix} 2 & 5 \end{bmatrix} \qquad D = 0$$

2. From Difference Equation

e.x.

$$y[(k+3)T] + 4y[(k+2)T] + 6y[(k+1)T] + 7y(kT) = 5v(kT)$$
$$y[(k+3)T] = -4y[(k+2)T] - 6y[(k+1)T] - 7y(kT) + 5v(kT)$$
$$y(kT) = -4y[(k-1)T] - 6y[(k-2)T] - 7y[(k-3)T] + 5v[(k-3)T]$$



define shift operation E

$$Ef(kT) \overset{\Delta}{=} f[(k+1)T]$$
$$E \cdot Ef(kT) = f[(k+2)T]$$
$$\frac{1}{E} f(kT) = f[(k-1)T]$$

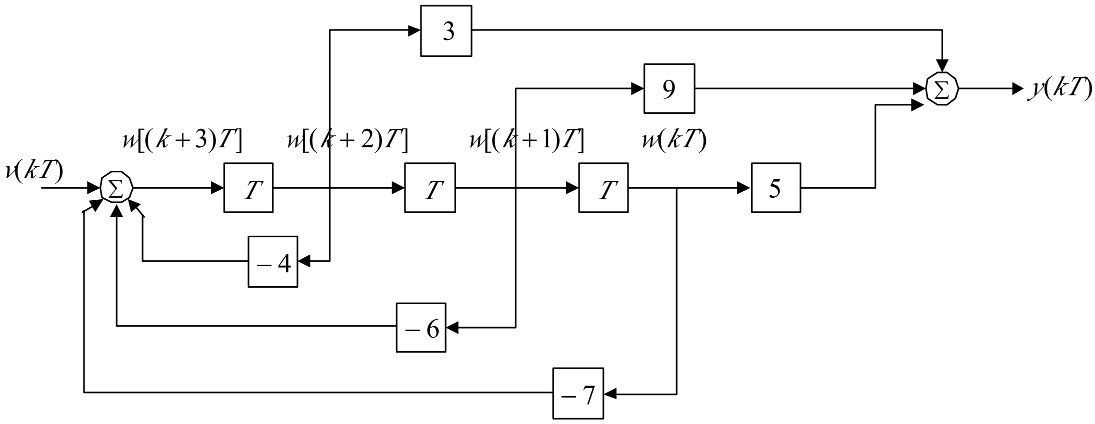
$$E^3 y(kT) + 4E^2 y(kT) + 6Ey(kT) + 7y(kT) = (3E^2 + 9E + 5)v(kT)$$

Method 1. $$\frac{y(kT)}{v(kT)} = \frac{3E^2 + 9E + 5}{E^3 + 4E^2 + 6E + 7}$$

Define $$\frac{v(kT)}{E^3 + 4E^2 + 6E + 7} = w(kT)$$

$$v(kT) = (E^3 + 4E^2 + 6E + 7)w(kT)$$

$$y(kT) = (3E^2 + 9E + 5)w(kT)$$



Method 2. nested form

$$E^3 y(kT) = -4E^2 y(kT) - 6Ey(kT) - 7y(kT) + 5v(kT)$$

$$y(kT) = \frac{1}{E}\{-4y(kT) + \frac{1}{E}[-6y(kT) + \frac{1}{E}(-7y(kT) + 5v(kT))]\}$$

3. From Approximate Continuous System State Equation

这 部 分 内 容 上 课 的 时 候 没 有 讲？ ？ ？ ？ ？ ？

Solution of the unforced system

Given $\begin{cases} X(k+1) = AX(k) \\ Y = CX(k) \end{cases}$ $X(0)$ find $X(k), Y(k)$

Solution : $k = 0 \quad X(1) = AX(0)$

$$X(2) = AX(1) = A^2 X(0)$$

$$X(3) = AX(2) = A^3 X(0)$$

$$\vdots$$

$$X(k) = A^k X(0)$$

A^K state transition matrix, i .e. $\phi(k,0)$

Methods for finding A^k

1. by direct multiplication

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \text{ diagonal matrix } \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^{\ k} & & \\ & \ddots & \\ & & \lambda_n^{\ k} \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda_1 & 1 & \\ & \ddots & 1 \\ & & \lambda_n \end{bmatrix} \text{ Jordan form}$$

Binomial Theorem: scalars $(a+b)^2 = a^2 + 2ab + b^2$

$$(a+b)^n = a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \cdots$$

matrices $(A+B)^2 = (A+B)(A+B)$

$$= A^2 + BA + AB + B^2$$

$$= A^2 + 2AB + B^2 \quad \text{iff } AB = BA$$

$$(A+B)^n = A^n + C_n^1 A^{n-1} B + C_n^2 A^{n-2} B^2 + \cdots$$

iff $AB = BA$

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}^k = \left\{ \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \right\}^k$$

$$= \begin{bmatrix} \lambda^k & & & \\ & \lambda^k & & \\ & & \lambda^k & \\ & & & \lambda^k \end{bmatrix} + k \begin{bmatrix} 0 & \lambda^{k-1} & & \\ & 0 & \lambda^{k-1} & \\ & & 0 & \lambda^{k-1} \\ & & & 0 \end{bmatrix} + \frac{k(k-1)}{2!} \begin{bmatrix} 0 & 0 & \lambda^{k-2} & \\ & 0 & 0 & \lambda^{k-2} \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} + \cdots$$

2. diagonalize

$M^{-1}AM = \Lambda$ or J

$$A^k = (M\Lambda M^{-1})(M\Lambda M^{-1})\cdots(M\Lambda M^{-1}) = M\Lambda^k M^{-1}$$

Ex. $X(k+1) = \begin{bmatrix} 0 & 1 \\ -0.5 & 1.5 \end{bmatrix} X(k)$, $X(0)$ find $X(k)$

Solution: $X(k) = A^k X(0)$ $\quad A = \begin{bmatrix} 0 & 1 \\ -0.5 & 1.5 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 0.5 & \lambda - 1.5 \end{vmatrix} = \lambda^2 - 1.5\lambda + 0.5 = (\lambda - 0.5)(\lambda - 1)$$

$$\lambda_1 = 0.5 \ , \ \lambda_2 = 1$$

$$M = [M_1, M_2] \quad \begin{cases} [\lambda_1 I - A]M_1 = 0 \\ [\lambda_2 I - A]M_2 = 0 \end{cases}$$

$$M = \begin{bmatrix} 1 & 1 \\ 0.5 & 1 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 1 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.5^k & 0 \\ 0 & 1^k \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 + 2\times 0.5^k & 2 - 2\times 0.5^k \\ 1 + 0.5^k & 2 - 0.5^k \end{bmatrix}$$

2. Caylay-Hamilton Theorem

Every square matrix satisfies its own characteristic equation

$A = \begin{bmatrix} 0 & 1 \\ -0.5 & 1.5 \end{bmatrix}$ find A^k

$$\lambda^k = ?1 + ?\lambda = \alpha_0(k) + \alpha_1(k)\lambda$$

$$A^k = \alpha_0(k) + \alpha_1(k)A$$

$$\begin{cases} 0.5^k = \alpha_0(k) + \alpha_1(k)\cdot 0.5 \\ 1^k = \alpha_0(k) + \alpha_1(k) \end{cases} \qquad \begin{cases} \alpha_0(k) = -1 + 2\times 0.5^k \\ \alpha_1(k) = 2 - 2\times 0.5^k \end{cases}$$

$$A^k = (-1 + 2\times 0.5^k)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (2 - 2\times 0.5^k)\begin{bmatrix} 0 & 1 \\ -0.5 & 1.5 \end{bmatrix}$$

4. Z transformation

Given $\begin{cases} X(k+1) = AX(k) + BV(k) \\ Y(k) = CX(k) + DV(k) \end{cases}$ $X(0)$

find $X(k)$ and $Y(k)$ for $k = 0,1,2,\cdots$

solution: $X(0)$

$$k = 0, X(1) = AX(0) + BV(0)$$

$$k = 1, X(2) = AX(1) + BV(1) = A^2X(0) + ABV(0) + BV(1)$$

$$k = 2 \ X(3) = AX(2) + BV(2) = A^3X(0) + A^2BV(0) + ABV(1) + BV(2)$$

In general $X(k) = A^kX(0) + A^{k-1}BV(0) + A^{k-2}BV(1) + \cdots + A^0BV(k-1)$

$$= A^kX(0) + \sum_{n=0}^{k-1} A^{k-1-n}u(k-1)BV(n)$$

for $k = 0,1,2,\cdots$ here $u(\cdot)$ is unit function

$$Y(k) = CA^kX(0) + \sum_{n=0}^{k-1} CA^{k-1-n}u(k-1)BV(n) + DV(k)$$

$$f(k) \quad Z\{f(k)\} \stackrel{\Delta}{=} \sum_{k=0}^{\infty} f(k)\frac{1}{z^k} = F(z)$$

$$= f(0) + f(1)\frac{1}{z} + f(2)\frac{1}{z^2} + \cdots + f(k)\frac{1}{z^k}$$

$$f(t) \quad Z\{f(kT)\} \stackrel{\Delta}{=} \sum_{k=0}^{\infty} f(kT)\frac{1}{z^k} = F(z)$$

$$= f(0) + f(T)\frac{1}{z} + f(2T)\frac{1}{z^2} + \cdots + f(kT)\frac{1}{z^k}$$

Ex. $f(t) = u(t)$ find $F(z)$

$$F(z) = 1 + 1\cdot\frac{1}{z} + 1\cdot\frac{1}{z^2} + \cdots + 1\cdot\frac{1}{z^k} + \cdots$$

$$= \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1} \qquad \text{iff } \left|\frac{1}{z}\right| < 1$$

Ex. $f(t) = e^{-at}$ $t = kT$ T is sampling period

$$F(z) = 1 + e^{-aT}\cdot\frac{1}{z} + e^{-2aT}\cdot\frac{1}{z^2} + \cdots + e^{-akT}\cdot\frac{1}{z^k} + \cdots$$

$$= \frac{1}{1-\frac{1}{e^{aT}z}} = \frac{z}{z-e^{-aT}} \qquad \text{iff } \left|\frac{1}{e^{aT}z}\right| < 1$$

Ex. $f(t) = t$

$$F(z) = 0 + T\cdot\frac{1}{z} + 2T\cdot\frac{1}{z^2} + \cdots + kT\cdot\frac{1}{z^k} + \cdots$$

$$= T\cdot\frac{1}{z}[1 + 2\cdot\frac{1}{z} + 3\cdot\frac{1}{z^2} + \cdots + k\cdot\frac{1}{z^{k-1}} + \cdots]$$

$$(1 + w + w^2 + w^3 + \cdots)(1 + w + w^2 + w^3 + \cdots)$$
$$= 1 + 2w + 3w^2 + 4w^3 + \cdots$$

$$T\cdot\frac{1}{z}[1 + 2\cdot\frac{1}{z} + 3\cdot\frac{1}{z^2} + \cdots + k\cdot\frac{1}{z^{k-1}} + \cdots]$$
$$= \frac{T}{z}(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots)(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots)$$
$$= \frac{T}{z}\left(\frac{1}{1-\frac{1}{z}}\right)^2 = \frac{Tz}{(z-1)^2}$$

$$\text{iff } \left|\frac{1}{z}\right| < 1$$

$$f(t) \leftrightarrow F(z) \quad f(t+T) \leftrightarrow zF(z) - zf(0)$$

proof: $Z\{f(t+T)\} = z \cdot \frac{1}{z}\{f(T) + \frac{1}{z} f(2T) + \frac{1}{z^2} f(3T) + \cdots\}$

$$= z\{\frac{1}{z} f(T) + \frac{1}{z^2} f(2T) + \frac{1}{z^3} f(3T) + \cdots\}$$

$$= z\{f(0) + \frac{1}{z} f(T) + \frac{1}{z^2} f(2T) + \frac{1}{z^3} f(3T) - f(0) + \cdots\}$$

$$= zF(z) - zf(0)$$

Inverse Z Transform

1. Long division

$$F(z) = \frac{4z^5 + 2z + 1}{z^6 + 9z + 1} = 4\frac{1}{z} + 2\frac{1}{z^5} + \cdots$$

$$\begin{array}{r}
4\frac{1}{z} + 2\frac{1}{z^5} \qquad\qquad \\
z^6 + 9z + 1 \overline{\big) 4z^5 + 2z + 1} \\
4z^5 \qquad + 36 + 4\frac{1}{z} \\
\hline
2z - 35 - 4\frac{1}{z} \\
2z \qquad\qquad + 18\frac{1}{z^4} + 2\frac{1}{z^5} \\
\hline
-35 - 4\frac{1}{z} - 18\frac{1}{z^4} - 2\frac{1}{z^5}
\end{array}$$

$$f(k) = \begin{cases} 0 & k = 0 \\ 4 & k = 1 \\ 0 & k = 2,3,4 \\ 2 & k = 5 \\ \vdots & \vdots \end{cases}$$

$$F(z) = \frac{z}{z-a} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots = \frac{1}{1 - \frac{a}{z}} \qquad |z| > a, a > 0$$

$$f(k) = a^k$$

reference formula

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots = \sum_{k=0}^{\infty} w^k$$

$$\frac{1}{(1-w)^2} = (1 + w + w^2 + \cdots)(1 + w + w^2 + \cdots)$$

$$= 1 + 2w + 3w^2 + 4w^3 + \cdots + (k+1)w^k + \cdots$$

$$= \sum_{k=0}^{\infty} (k+1) w^k$$

$$\frac{1}{(1-w)^3} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2!} w^k$$

$$\frac{1}{(1-w)^5} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)(k+3)(k+4)}{4!} w^k$$

$F(z) = \dfrac{z}{(z-a)^2}$ find $f(k)$

$$F(z) = \frac{1}{z} \cdot \frac{1}{(1-\frac{a}{z})^2} = \frac{1}{z}(1 + 2\frac{a}{z} + 3(\frac{a}{z})^2 + \cdots) = \frac{1}{z} + 2\frac{a}{z^2} + 3\frac{a^2}{z^3} + \cdots$$

2. Partial Fraction Expansion

Ex. $F(z) = \dfrac{2z+7}{(z-\frac{1}{3})(z-\frac{1}{5})(z-\frac{1}{7})}$

$$\frac{F(z)}{z} = \frac{1}{z} \cdot \frac{2z+7}{(z-\frac{1}{3})(z-\frac{1}{5})(z-\frac{1}{7})}$$

$$F(z) = b_0 + b_1 \frac{z}{z-\frac{1}{3}} + b_2 \frac{z}{z-\frac{1}{5}} + b_3 \frac{z}{z-\frac{1}{7}}$$

$$f(k) = b_0 \delta(k) + b_1 (\frac{1}{3})^k + b_2 (\frac{1}{5})^k + b_3 (\frac{1}{7})^k$$

the 1st term = $\begin{cases} b_0 & k=0 \\ 0 & k=1,2,3,\cdots \end{cases}$

3. Residue Theorem

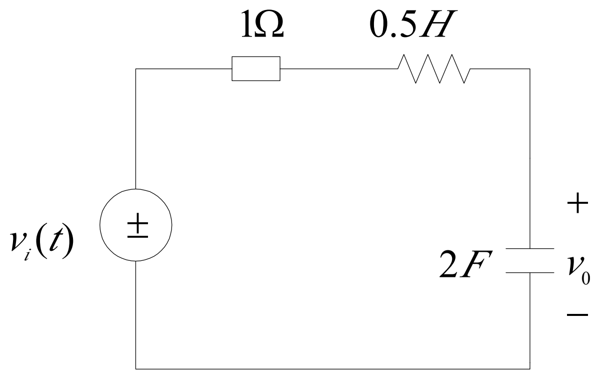
$$F(z)=\frac{z}{z^4-1}=\frac{z}{(z^2+1)(z+1)(z-1)}$$

$$=\frac{1}{z^3}\cdot\frac{1}{1-\frac{1}{z^4}}=\frac{1}{z^3}\cdot(1+\frac{1}{z^4}+\frac{1}{z^8}+\frac{1}{z^{12}}+\cdots)$$

$$=\frac{1}{z^3}+\frac{1}{z^{11}}+\frac{1}{z^{15}}+\cdots$$

$$f(k)=\begin{cases} 1 & k=3,7,11,15,\cdots \\ 0 & otherwise \end{cases}$$

Part 3. Sinusoidal Steady State Analysis



Given $v_i(t)=10\cos 2t$

Find $v_0(t)$ is steady state

$$H(s)=\frac{v_0}{v_i}=\frac{1}{s^2+2s+1}$$

$$v_i(s)=\frac{10s}{s^2+4}$$

$$v_0(s)=\frac{1}{s^2+2s+1}\cdot\frac{10s}{s^2+4}=\frac{10s}{(s+1)^2(s^2+4)}$$

let $s = jw = j2 \quad H(j2) = \frac{1}{-3+4j} = \frac{1}{5\angle 2.21}$

$$v_0 = 2e^{-j2.21}$$

$$v_0(t)|_{t=\infty} = 2\cos(2t - 2.21)$$

Ex. $\frac{v_0}{v_i} = H(s) = \frac{s^2+s+1}{s^3+0.5s^2+0.5s+1}$

$$v_i = \cos t, \quad w = 1, \quad s = jw = j1$$

find $v_0(t)$ in steady state

solution: $H(j1) = -1 + j = \sqrt{2}e^{j\frac{3}{4}\pi}$

$$v_0(t)\big|_{t\to\infty} = \sqrt{2}\cos(t + \frac{3\pi}{4})$$

check whether $v_0(t)\big|_{t\to\pi}$ is right or not .

$$v_0 = H(s)v_i(s) = \frac{s^2+s+1}{s^3+0.5s^2+0.5s+1} \cdot \frac{s}{s^2+1}$$

poles: $s = -1, s = \frac{0.5 \pm j\sqrt{0.25-4}}{2}$

so the system is unstable, cann't solve this question by the method: $v_0 = Hv_i$

discrete: Given $\begin{cases} X[(k+1)T] = AX(kT) + BV(kT) \\ Y(kT) = CX(kT) + DV(kT) \end{cases}$

$$V(kT) = V_m\cos(kwT)$$

find $Y(kT)$ in steady state

$$Y(z) = C[zI-A]^{-1}zX(0) + \{C[zI-A]^{-1}B + D\}V(z)$$

assume the 1st term contributes nothing to steady state.
Because eigenvalues of $A < 1$
Then $Y(z) = H(z)V(z)$

$$V(kT) = v_m \cos kwT = v_m \frac{e^{jkwT} + e^{jkwT}}{2}$$

$$V(z) = \frac{v_m}{2} \frac{z}{z - e^{jwT}} + \frac{v_m}{2} \frac{z}{z - e^{-jwT}}$$

$$Y(z) = H(z)[\frac{v_m}{2} \frac{z}{z - e^{jwT}} + \frac{v_m}{2} \frac{z}{z - e^{-jwT}}]$$

$$= \frac{wz}{z - e^{jwT}} + \frac{\overline{w}z}{z - e^{-jwT}} + \cdots \Rightarrow 0$$

iff. $|pole| < 1$

Ex. $H(z) = \dfrac{z}{(z - \frac{1}{2})(z - \frac{1}{4})}$ in steady state

$$Y(z) = \frac{wz}{z - e^{jwT}} + \frac{\overline{w}z}{z - e^{-jwT}}$$

where $w = \lim_{z \to e^{jwT}} [\frac{Y(z)}{z}(z - e^{jwT})] = \frac{v_m}{2} H(e^{jwT}) = \frac{v_m}{2} H_m e^{j\theta}$

Then $Y(z) = \dfrac{v_m H_m e^{j\theta}}{2} \dfrac{z}{z - e^{jwT}} + \dfrac{v_m H_m e^{-j\theta}}{2} \dfrac{z}{z - e^{-jwT}}$

$$Y(kT) = \frac{v_m H_m e^{j\theta}}{2} e^{jkwT} + \frac{v_m H_m e^{-j\theta}}{2} e^{-jkwT}$$

$$= \frac{v_m H_m}{2} (e^{j(\theta + kwT)} + e^{-j(\theta + kwT)})$$

$$= v_m H_m \cos(kwT + \theta)$$

Ex. $\dfrac{Y}{V} = H(z) = \dfrac{z}{z - \frac{1}{2}}$

$$V(kT) = 10\cos(wt + \frac{\pi}{4}) = 10\cos(0.2k + \frac{\pi}{4})$$

if $w = 2 \quad T = 0.1$

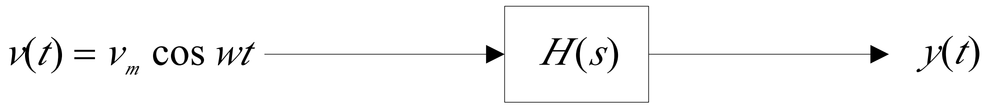
find $Y(kT)$ in steady state

solution: $[H(z)]_{z=e^{j0.2}} = \dfrac{e^{j0.2}}{e^{j0.2}-\dfrac{1}{2}} = 1.925e^{-j0.19}$

$$Y(kT) = 19.25\cos(2kT + \frac{\pi}{4} - 0.19) = 19.25\cos(0.2k + 0.595)$$

conclusion:

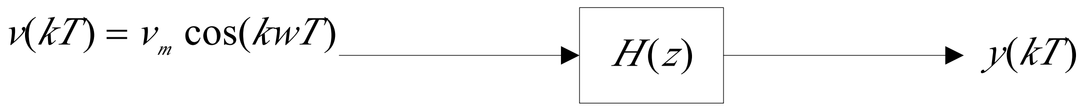
1. continuous



$y(t) = H_m v_m \cos(wt + \theta)$ where $H_m e^{j\theta} = [H(s)]_{s=jw}$

if the system is stable.

2. discrete



$y(kT) = H_m v_m \cos(kwT + \theta)$ where $H_m e^{j\theta} = [H(z)]_{z=e^{jwT}}$

Chapter Ⅱ Lyapunov Stablility

Section 1 Stability Definition

1. Asymptotic Stability (zero input)

Without any input, $\dot{X}, X$ eventually become zero

1) continuous system $\begin{cases} \dot{X}(t) = AX + BV \\ Y(t) = CX + DV \end{cases}$

The system is said to be asymptotic stable with zero input, $\lim_{t\to\infty} X(t) = 0$ for all $X(0)$.

2) discrete system $\begin{cases} X(k+1) = AX(k) + BV(k) \\ Y(k) = CX(k) + DV(k) \end{cases}$

The system is said to be asymptotic stable with zero input, $\lim_{t\to\infty} X(k) = 0$ for all $X(0)$.

Testing method:

1) continuous system

$$X(t) = e^{At} X(0)$$

$$e^{At} = L^{-1}[sI - A]^{-1} = L^{-1} \frac{adj[sI - A]}{|sI - A|}$$

$$|sI - A| = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

All the eigenvalues of A should have negative real part $\Leftrightarrow$ The system is asymptotic stable.

2) discrete system

$$X(k) = A^k X(0)$$

$$A^k = z^{-1}\{[zI - A]^{-1} z\} = z^{-1}\{\frac{adj[zI - A]}{|zI - A|}\} = z^{-1}\{\frac{adj[zI - A]}{(z - \lambda_1)\cdots(z - \lambda_n)}\}$$

All the eigenvalues of A have magnitude less than 1 (within unit circle) $\Leftrightarrow$ The system is asymptotic stable.

2. BIBO—bounded input bounded output (zero state)

The system is said to be BIBO stable if with $X(0) = 0$

Bounded inputs bounded outputs:

1) all the poles of transfer function $H(s)$ have negative real part $\Leftrightarrow$ BIBO stable.

$$Y(s) = C[sI - A]^{-1} X(0) + \{C[sI - A]^{-1} B + D\} V(s)$$

$$Y(s) = H(s) V(s)$$

number of poles $\le$ number of eigenvalues of A

2) all the poles of $H(s)$ have magnitude less than 1.

$$H(s) = C[sI - A]^{-1} B + D = \frac{Cadj[sI - A]B + D[sI - A]}{|sI - A|}$$

conclusion: Asymptotic stable $\Rightarrow$ BIBO stable

BIBO stable $\Rightarrow$ may be or may be not Asymptotic stable

In steady state

$y(t) = H_m V_m \cos(wt + \theta)$ if system is AS or BIBO &

$X(0) = 0$.

$y(kT) = H_m V_m \cos(kwT + \theta)$, $H_m e^{j\theta} = [H(z)]_{z=e^{jwT}}$

if system is AS or BIBO & initially relaxed.

3. Lyapunov Stability

With $V(t)=0$,if for every $X(0), X(t)$ remains bounded for all t, then we say the system is stable in sense of Lyapunov.

Testing method: 1) Eigenvalues of A $\operatorname{Re}\lambda_i < 0$ for all λ_i, then stable in sense of Lyapunov.

2) if some $\operatorname{Re}\lambda_i > 0 \Rightarrow$ not stable in sense of Lyapunov.

3) If some λ_i has zero real part, need further work.

Section 2. Lyapunov Theorem

Positive Definition Quadratic Function

Let $v(x)$ is a scalor function of vector X, S is a closed finite area of X space including origin,

Definition: $v(x)$ is positive definite (or semi-positive definite) if for all the X in s:

1) $v(x)$ has continuous pariial derivatives to each element in X.

2) $V(0)=0$.

4) when $X \neq 0, v(x) > 0$ (or $v(x) \geq 0$)

if in 3) the inequality is in opposite direction, then $v(x)$ negative definite.

Ex. 1.function $v(x) = x_1^2 + x_2^2$

1) when $x_1 = x_2 = 0, v(x) = 0$

2) x_1 or $x_2 \neq 0, v(x) > 0$,therefore $v(x)$ is positive definite

2.function $v(x) = (x_1 + x_2)^2$

1) when $x_1 = x_2 = 0, v(x) = 0$

2) x_1 or $x_2 \neq 0,\ v(x) \geq 0$, therefore $v(x)$ is semi-positive definite.

The quadratic function $p(x) = X^T P X$

P : weighting matrix , $P = \begin{bmatrix} p_{11} & & p_{1n} \\ & \ddots & \\ p_{n1} & & p_{nn} \end{bmatrix}$

$$p(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij} x_i x_j$$

Sylvester Theorem

a. P is a symmetric matrix, $P(X)$ is positive definite, iff all the pre-sub-determinants are positive.

i.e. p_{11}, $\det\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $\det\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$, $\cdots, \det[P]$

b. if all the sub-determinants are non-negative, then $P(X)$ is semi-positive definite.

i.e. $p_{11}, p_{22}, p_{33}, \det\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \det\begin{bmatrix} p_{11} & p_{13} \\ p_{31} & p_{33} \end{bmatrix}, \det\begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix}, \cdots, \det[P]$

c. if $-P(X)$ is positive definite, the $P(X)$ is negative definite.

Lyapunov Theorem

System $\dot{X} = f(X,t)$ is stable at its origin, if there exists a positive definite function $V(X)$ in some neighborhood of the origin, such that the derivative of V, $\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(X,t)$, is semi-negative definite.

Furthermore, $\lim_{\|X\| \to \infty} V(X) \to \infty$, then $X_e = 0$ is stable in large range.

Another version: $V(X) > 0, \dot{V}(X) < 0$, then the system is AS.

The zero solution of system $\dot{X} = f(X,t)$ is unstable, if in some neighborhood R of origin there exists a positive definite function $V(X)$, its derivative $\dot{V}(X)$ is positive definite in R.

That is: $V(X) > 0, \dot{V}(X) > 0$, then zero solution of system $\dot{X} = f(X,t)$ is unstable.

Conclusion:① $V(X) > 0, \dot{V}(X) \le 0 \Rightarrow \textit{stable}$

② $V(X) > 0, \dot{V}(X) < 0 \Rightarrow AS$

③ $V(X) > 0, \dot{V}(X) > 0 \Rightarrow \textit{unstable}$

ex. $\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2) \end{cases}$

equilibrium state (condition: $\dot{x}_1 = \dot{x}_2 = 0$

i.e. $x_1 = 0, x_2 = 0$

define a Lyapunov Function as $V(X) = x_1^2 + x_2^2$

furthermore, $\dot{V}(X) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$$= 2x_1x_2 - 2x_1^2(x_1^2 + x_2^2) - 2x_1x_2 - 2x_2^2(x_1^2 + x_2^2)$$
$$= -2(x_1^2 + x_2^2)^2 < 0$$

then the system is AS.

ex. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

the only balance state: $x_1 = 0, x_2 = 0$

1. define Lyapunov Function as $V(X) = x_1^2 + x_2^2$ is positive definite,

furthermore, $\dot{V}(X) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2x_2^2$ is semi-negative definite ,
at the equilibrium state is stabe, not AS.
2. define another Lyapunov Function as $V(X) = \frac{1}{2}[(x_1 + x_2)^2 + 2x_1^2 + x_2^2)$ is positive definite,

furthermore, $\dot{V}(X) = (x_1 + x_2)(\dot{x}_1 + \dot{x}_2) + 2x_1\dot{x}_1 + x_2\dot{x}_2 = -(x_1^2 + x_2^2) < 0$ is negative definite,
so the balance state is AS.
Conclusion: ① Lyapunov Function is not unique;
② L.T. is sufficient condition of stability, not necessary condition.

Section 3. The Application of Lyapunov Method in Linear System

1. Test the stability of Linear System

System $\dot{X} = AX$, define Lyapunov Function $V(X) = X^T PX$,

The derivative of $V(X)$ to t: $\dot{V}(X) = \dot{X}^T PX + X^T P\dot{X}$

$$= X^T A^T PX + X^T PAX$$
$$= X^T (A^T P + PA)X = X^T QX$$

$$Q = A^T P + PA$$

if $-Q$ is positive definite, the Q is negative definite.

Let $Q=-I$, see P is positive definite or not?

Theorem: $\dot{X}=AX$ the eigenvalues of A λ_i,λ_j for all $i,j=1,2,\cdots,\lambda_i\neq\lambda_j\neq 0$,such that $Q=A^TP+PA$ has unique solution.

ex. $A=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$

let $P=\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}$, $Q=-I$

$$\begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}+\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$p_{11}=\frac{5}{4},p_{12}=p_{22}=\frac{1}{4}$$

$P=\begin{bmatrix}\frac{5}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4}\end{bmatrix}$, P is positive definite.

So X^TPX is positive definite.
So system is stable.

Method 1. find the Lyapunov Function

$V(X)=[x_1,x_2]\begin{bmatrix}\frac{5}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\frac{1}{4}\left[5x_1^2+2x_1x_2+x_2^2\right]$ is positive definite

$\dot{V}(X)=\frac{1}{4}[10x_1\dot{x}_1+2x_2\dot{x}_1+2x_1\dot{x}_2+2x_2\dot{x}_2]=-(x_1^2+x_2^2)$ is negative definite.

So system is stable in large range.

Method 2. Sylvester Theorem

$$p_{11}=\frac{5}{4}>0,\det P=\begin{vmatrix}\frac{5}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4}\end{vmatrix}=\frac{1}{4}>0$$

$\therefore P$ is positive definite.

$\therefore$ the origin is AS.

Ex. $\begin{cases}\dot{x}_1=-x_1-2x_2\\ \dot{x}_2=x_1-4x_2\end{cases}$ the equilibrium state: if $\dot{x}_1=\dot{x}_2=0$

$$\begin{cases} -x_1 - 2x_2 = 0 \\ x_1 - 4x_2 = 0 \end{cases} \qquad \therefore x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Given $A = \begin{bmatrix} 1 & -2 \\ 1 & -4 \end{bmatrix}$ let $Q = -I$

$$\therefore p_{11} = \frac{23}{60}, p_{12} = \frac{-7}{60}, p_{22} = \frac{11}{60}$$

$P = \begin{bmatrix} \frac{23}{60} & \frac{-7}{60} \\ \frac{-7}{60} & \frac{11}{60} \end{bmatrix}$ is positive definite.

$\therefore$ the equilibrium state is AS.

2. Several Performance indices of Lyapunov system

Define $\eta = -\frac{\dot{V}(X,t)}{V(X,t)}$ to AS system , $\eta > 0$

$\eta^{\uparrow}$, the convergence of the movement is faster.

If η is a constant, then $V(X,t) = V(X,t)e^{-\int_{t_0}^{t} \eta dt}$

$$= V(X_0,t)e^{\eta(t_0-t)}$$

define $\eta_{\min} = \min[-\frac{\dot{V}(X,t)}{V(X,t)}]$ is the slowest velocity towards the origin.

System $\dot{X} = AX$ find $\eta_{\min}$

All the eigenvalues have negative real part, i.e. the system is AS in large range.

Let $V = X^T PX$ then $\dot{V} = X^T QX$ where $Q = A^T P + PA$

Under the constraint of $V = X^T PX = 1$ to find X such that $\dot{V} = X^T QX$ is minimized, i.e. $\min\{X^T QX; X^T PX = 1\} = \min\left\{-\frac{\dot{V}(X,t)}{V(X,t)}\right\} = \eta_{\min}$

Solve by Lagrange Multiplier

$$N = X^T QX + \mu(1 - X^T PX)$$

$$\frac{\partial N}{\partial X} = 0 = 2QX - \mu 2PX\Big|_{X_{\min}} = 0$$

$$(Q-\mu P)X_{\min}=0$$

$$\det(Q-\mu P)=0$$

$$\left|QP^{-1}-\mu I\right|=0$$

ex. Given $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}, \xi=0.5, \varpi_n=1$

estimate the maximum time constants by Lyapunov Function

define $V=X^TPX$ then $\dot{V}=X^TQX=X^T(A^TP+PA)X$

let $Q=-I$ therefore $\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}+\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

$P=\begin{bmatrix}\frac{3}{2} & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}$, so $V=X^TPX=\frac{1}{2}[(x_1+x_2)^2+2x_1^2+x_2^2]$

$\dot{V}(X)=-(x_1^2+x_2^2)$, $\eta=-\frac{\dot{V}}{V}=\frac{2(x_1^2+x_2^2)}{3x_1^2+2x_1x_2+2x_2^2}$

$\frac{\partial\eta}{\partial x_1}=0$, then $x_1=1.618x_2, x_1=-0.618x_2$

substitute in the equation $\eta_{\min}=0.553, \eta_{\max}=1.447$

$\therefore$ the upper bound of V converging time is $\frac{1}{\eta_{\min}}=1.81s$

$\therefore$ $V(X)$ converges twice faster than $X(t)$, i.e. the upper bound of converging time for X, $\tau=\frac{2}{\eta_{\min}}=3.62s$

time constant $T=\frac{1}{\xi\varpi_n}=2s$

ex. Given the 2nd order system

$\ddot{x}+2\xi\dot{x}+x=0$ the damping coefficient is greater than zero.

Define $V(X,\xi) = \int_{t_0}^{t} X^T(t)QX(t)dt$

Let $Q = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \mu \geq 0$, find ξ to minimized V

$$\dot{X} = AX, \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2\xi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2\xi \end{bmatrix}$$

solve the equ. Above $P(\xi) = \begin{bmatrix} \xi + \frac{1+\mu}{4\xi} & \frac{1}{2} \\ \frac{1}{2} & \frac{1+\mu}{4\xi} \end{bmatrix}$

$$V(X(t_0)) = X^T(t_0)P + PX(t_0)$$

$$= \xi x_1^2(t_0) + [\frac{1+\mu}{4\xi}][x_1^2(t_0) + x_2^2(t_0)] + x_1(t_0)x_2(t_0)$$

when $x_2(t_0) = 0$, then $\frac{\partial V(X(t_0))}{\partial \xi} = x_1^2(t_0)(1 - \frac{1+\mu}{4\xi^2})$,i.e. $\xi = \frac{\sqrt{1+\mu}}{2}$

choose $\mu = 0$,$\xi = \frac{1}{2}$, ξ is too small and system oscillates.

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, integration index $V(X,\xi) = \int x_1^2 dt$;

choose $\mu = 1, \xi = 0.707$, ξ is ideal damping.

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, integration index $V(X,\xi) = \int (x_1^2 + x_2^2)dt$

Section 4. The Application of Lyapunov Method in non-linear system

1. The 1st Lyapunov Method

Suppose $f(X)$ is continuous and differentiable vector function, it is linearized in the neighborhood of the equilibrium state to σX , the $\dot{X} = AX + g(0, X)$, where $A = (\frac{\partial f_i}{\partial x_i})_M \quad i, j = 1,2,3,\cdots n$ is Jacobian Matrix at point M.

The linear part of the equation $\sigma \dot{X} = A\sigma X$ is the 1st order approximation:

1) if the linearized equation is AS, then the original system is AS.
2) if the linearized equation has even one root with oppositive real part, then the zero solution of the original system is also unstable at the origin.
3) if the linearized eqution has some eigenvalues with zero real part (marginal case), then the stability of original non-linear system could not be tested by the linearized eqution. And the 2nd and higher order term must be considered and the 2nd Lyapunov Method is used.

Proof: define $V = X^T P X$, $\dot{V} = X^T X$, $Q = -I$

Derivative $\dot{V}(X) = -X^T X + (\sum_{\alpha,\beta=1}^{n} \frac{1}{2} \frac{\partial f_i(k_i X)}{\partial x_\alpha x_\beta}) x_\alpha x_\phi , 0 < k_i < 1$

The coefficients of 2nd order terms are bounded, therefore, in a sufficient neighborhood.

$$\dot{V} = -\|X\|^2 + \frac{1}{2}\|X\|^2 = -\frac{1}{2}\|X\|^2$$

obviously $\dot{V}$ is negative definite.

$\therefore$ original system is stable.

Ex. Given non-linear system
$$\begin{cases} \dot{x}_1 = -x_2 + a x_1^3 \\ \dot{x}_2 = x_1 + a x_2^3 \end{cases}$$

Test the stability at zero state solution.

$$A = \begin{bmatrix} 3ax_1^2 & -1 \\ 1 & 3ax_2^2 \end{bmatrix}_{\substack{x_1=0 \\ x_2=0}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

the characteristic equation and its eigenvalues:

$$D(\lambda) = |A - \lambda I| = \lambda^2 + 1 = 0, \lambda = \pm j$$

define Lyapunov Function $V = \frac{1}{2}(x_1^2 + x_2^2)$, then $\dot{V} = x_1(x_2 + a x_1^3) + x_2(x_1 + a x_2^3)$

$$= a(x_1^4 + x_2^4)$$

1) $a > 0$, $\dot{V}$ p.d., the origin is unstable.

2) $a = 0$, $\dot{V} = 0$, origin is stable, but not AS.

3) $a < 0$, $\dot{V}$ n.d., origin is AS.

2. Define Lyapunov Function for a non-linear system

no unified method for non-linear system

for pendulum

$$H = \frac{d^2\varphi}{dt^2} = -mg\sin\varphi \cdot l \quad \text{, where } \quad H = ml^2$$

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\sin\varphi$$

set $\varphi = x_1$, $\frac{d\varphi}{dt} = x_2$

then $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 \end{cases}$

let $V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{g}{l}(1 - \cos x_1)$, obviously $0 \le x_1 \le \pi$

so V is positive definite.

$$\dot{V} = x_2\dot{x}_2 + \frac{g}{l}\sin x_1 \dot{x}_1 = x_2(-\frac{g}{l}\sin x_1) + \frac{g}{l}\sin x_1 \cdot x_2 = 0$$

so system is stable at origin, but not AS.

Non-linear system $\begin{cases} \dot{x}_1 = -2x_1 + 2x_2^4 \\ \dot{x}_2 = -x_2 \end{cases}$

(a) 1st method

$$\frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2 & 8x_2^2 \\ 0 & -1 \end{bmatrix}_{X=\begin{bmatrix}0\\0\end{bmatrix}} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

linearized system $\dot{Y} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} Y$ at $Y = 0$

$$\therefore \lambda_1 = -2 < 0, \lambda_2 = -1 < 0$$

$\therefore$ linear system is AS at origin, also original system is AS at origin.

(b) Krasovski Method to build $V(X)$

Non-linear, time-invariant unforced system

$f(0) = 0$, $f(X)$ is differentiable to x_1, x_2

Jacobian Matrix $Q(X) = J^T(X) + J(X)$ (where $P = I$)

For the example above,

$$Q(X) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}$$ is negative definite.

Then $X_e = 0$ is AS (sufficient condition)

If $\|X\| \to \infty$, $V(X) = f^T(X) f(X) = [-2x_1 + 2x_2^4, -x_2] \begin{bmatrix} -2x_1 + 2x_2^4 \\ -x_2 \end{bmatrix}$

$$= (-2x_1 + 2x_2^4)^2 + x_2^2 \to \infty$$

then the system is globally AS.

(c) variable gradient method

set $V(X)$, $gradV = \begin{bmatrix} \dfrac{\partial V}{\partial x_1} \\ \dfrac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \nabla_1 \\ \nabla_2 \end{bmatrix}$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = (gradV)^T(\dot{X})$$

$$\frac{\partial(gradV)}{\partial X} = \begin{bmatrix} \dfrac{\partial \nabla_1}{\partial x_1} & \dfrac{\partial \nabla_1}{\partial x_2} \\ \dfrac{\partial \nabla_2}{\partial x_1} & \dfrac{\partial \nabla_2}{\partial x_2} \end{bmatrix}$$

if symmetric, the $rot(gradV) = 0$.

Step 1. let $gradV = \begin{bmatrix} a_{11}x_1 & a_{12}x_2 \\ a_{21}x_1 & a_{22}x_2 \end{bmatrix}$

Step 2. build $\dot{V}(X) = (a_{11}x_1 + a_{12}x_2)(-2x_1 + 2x_2^4) + (a_{21}x_1 + a_{22}x_2)(-x_2) < 0$

Step 3. from $gradV$ to build $V(X)$ in a conservative field

$$V(X) = \int_0^{x_1(x_2=0)} (a_{11}x_1 + a_{12}x_2)dx_1 + \int_0^{x_2(x_1=x_1)} (a_{21}x_1 + a_{22}x_2)dx_2$$

$$= \frac{1}{2}a_{11}x_1^2 + a_{21}x_1x_2 + \frac{1}{2}a_{22}x_2^2$$

Chapter III Canonical State Models

Controllable and Observable

Section 1. Canonical State Models

Four Canonical State Models

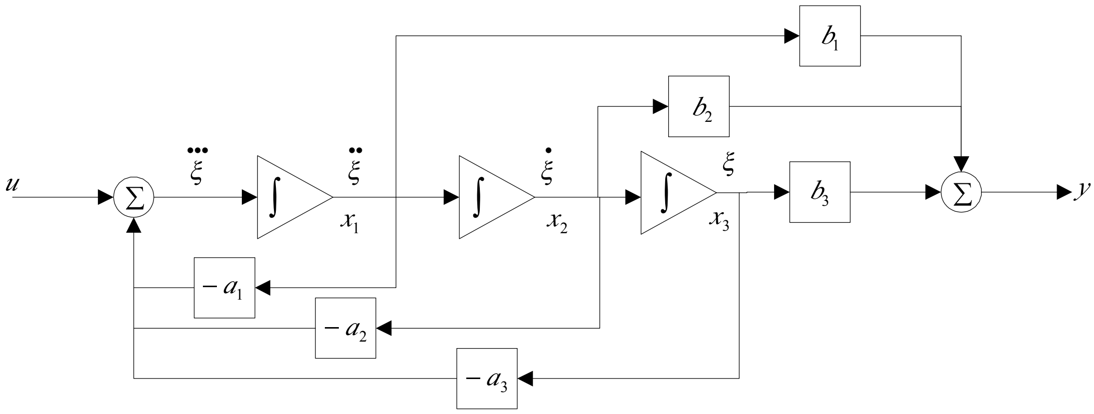
Ex. $\ddot{y} + 5y = u + 2\dot{u}$

$$\ddot{y} = F(\dot{y}, y, u, t) = -5y + u + 2\dot{u}$$

1) $\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_3 u + b_2\dot{u} + b_1\ddot{u}$

$$\frac{y}{u} = \frac{b_3 + b_2 p + b_1 p^2}{p^3 + a_1 p^2 + a_2 p + a_3} \cdot \frac{\xi}{\xi} \qquad x(0) = 0$$

$$u = (p^3 + a_1 p^2 + a_2 p + a_3)\xi$$
$$y = (b_3 + b_2 p + b_1 p^2)\xi$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

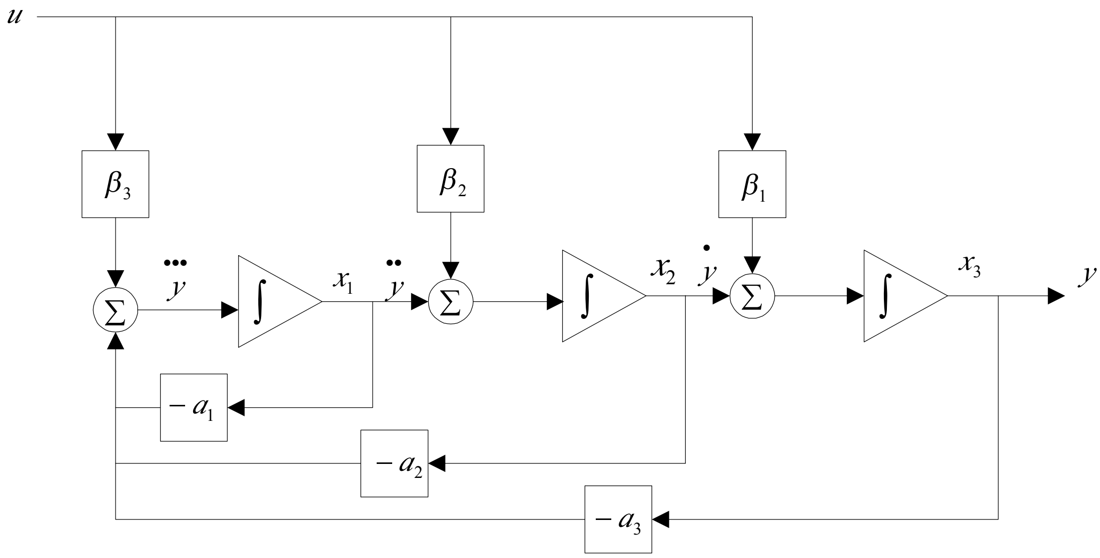
2) Laplace Transformation

$$(s^3 + a_1 s^2 + a_2 s + a_3)Y(s) = (b_1 s^2 + b_2 s + b_3)u(s)$$

$$Y(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} u(s) = \frac{b(s)}{a(s)} u(s)$$

$$Y(s) = b(s)a^{-1}(s)u(s) = b(s)\xi(s)$$

where $\xi(s) = a^{-1}(s)u(s)$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Let $x_1 = y \quad y = \frac{1}{p}(b_1 u - a_1 y) + \frac{1}{p^2}(b_2 u - a_2 y) + \frac{1}{p^3}(b_3 u - a_3 y)$

Define $\dot{x}_1 = py = b_1 u + x_2$ when $x_2 = -a_1 x_1 + \frac{1}{p}(b_2 u - a_2 x_1) + \frac{1}{p^2}(b_3 u - a_3 x_1)$

$$\dot{x}_2 = -a_1 \dot{x}_1 + (b_2 u - a_2 x_1) + \frac{1}{p}(b_3 u - a_3 x_1)$$

$$= (b_2 - a_1 b_1)u - a_1 x_2 - a_2 x_1 + \frac{1}{p}(b_3 u - a_3 x_1)$$

where $x_3 = -a_1 x_2 - a_2 x_1 + \frac{1}{p}(b_3 u - a_3 x_1)$

$$\dot{x}_3 = -a_1 \dot{x}_2 - a_2 \dot{x}_1 + b_3 u - a_3 x_1$$

$$= (a_1^2 b_1 - a_1 b_2 - a_2 b_1 + b_3)u - a_1 x_3 - a_2 x_2 - a_3 x_1$$

$$\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix}=\begin{bmatrix}b_1\\ -a_1b_1+b_2\\ a_1^2b_1-a_1b_2-a_2b_1+b_3\end{bmatrix}=\begin{bmatrix}1&0&0\\ -a_1&1&0\\ a_1^2-a_2&a_1&1\end{bmatrix}\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}=\begin{bmatrix}1&0&0\\ a_1&1&0\\ a_2&a_1&1\end{bmatrix}^{-1}\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$

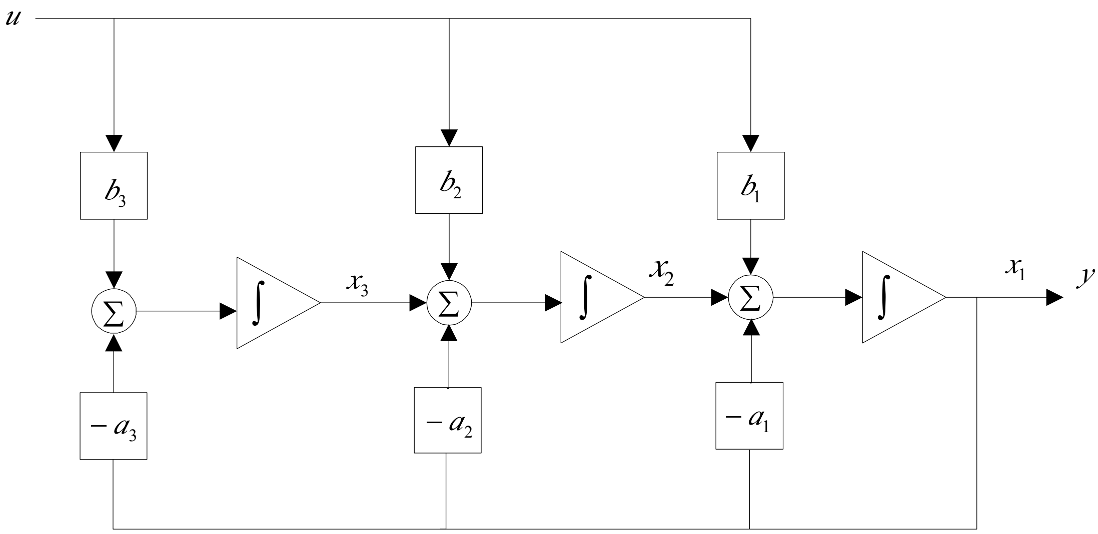
here β_1,β_2,β_3 Markov 数

3.Obvservable Canonical Form

$$y(p^3+a_1p^2+a_2p+a_3)=(b_3+b_2p+b_1p^2)u$$

$$y=-a_1\frac{y}{p}-a_2\frac{y}{p^2}-a_3\frac{y}{p^3}+\frac{b_3}{p^3}u+\frac{b_2}{p^2}u+\frac{b_1}{p}u$$

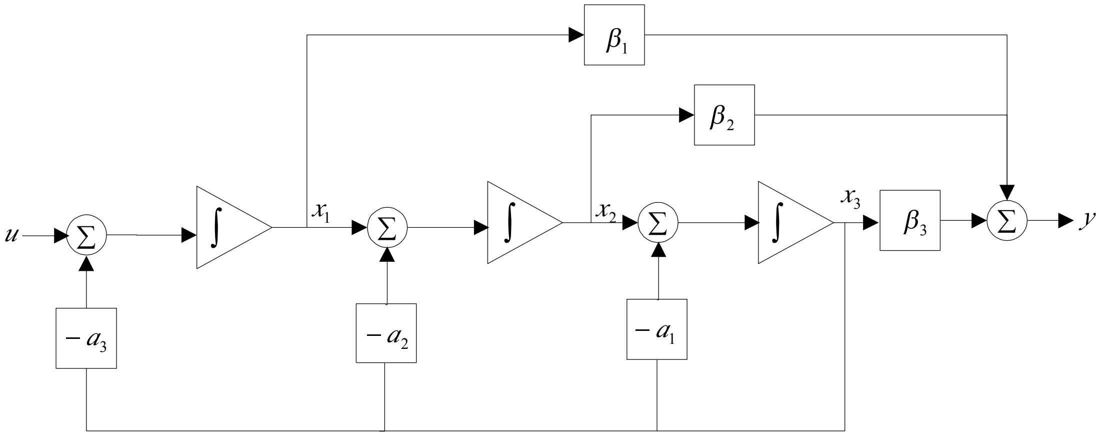
$$=\frac{1}{p}\{(b_1u-a_1y)+\frac{1}{p}[(b_2u-a_2y)+\frac{1}{p}(b_3u-a_3y)]\}$$



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-a_1&1&0\\ -a_2&0&1\\ -a_3&0&0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}u$$

$$y=\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+[0]u$$

4．Controllability Canonical Form



superposition $\frac{1}{p}\{-a_1x_3+\frac{1}{p}[-a_2x_3+\frac{1}{p}(u-a_3x_3)]\}=x_3$

Let $\hat{y}=x_3$

$\therefore \dddot{\hat{y}}+a_1\ddot{\hat{y}}+a_2\dot{\hat{y}}+a_3\hat{y}=u$

$y=(b_3+b_2p+b_1p^2)\hat{y}$

$y=b_3x_3+b_2(x_2-a_1x_3)+b_1p(x_2-a_1x_3)$

$$\begin{aligned}
&=b_3x_3+b_2(x_2-a_1x_3)+b_1\dot{x}_2-b_1a_1\dot{x}_3\\
&=b_3x_3+b_2(x_2-a_1x_3)+b_1(x_1-a_2x_3)-b_1a_1(x_2-a_1x_3)\\
&=(b_3-a_1b_2-a_2b_1+b_1a_1^2)x_3+(b_2-b_1a_1)x_2+b_1x_1
\end{aligned}$$

$$\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix}=\begin{bmatrix}b_1\\ -a_1b_1+b_2\\ a_1^2b_1-a_1b_2-a_2b_1+b_3\end{bmatrix}=\begin{bmatrix}1&0&0\\ -a_1&1&0\\ a_1^2-a_2&a_1&1\end{bmatrix}\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}=\begin{bmatrix}1&0&0\\ a_1&1&0\\ a_2&a_1&1\end{bmatrix}^{-1}\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$